Closed form solution of one-dimensional seismic wave equations with spatially inhomogeneity

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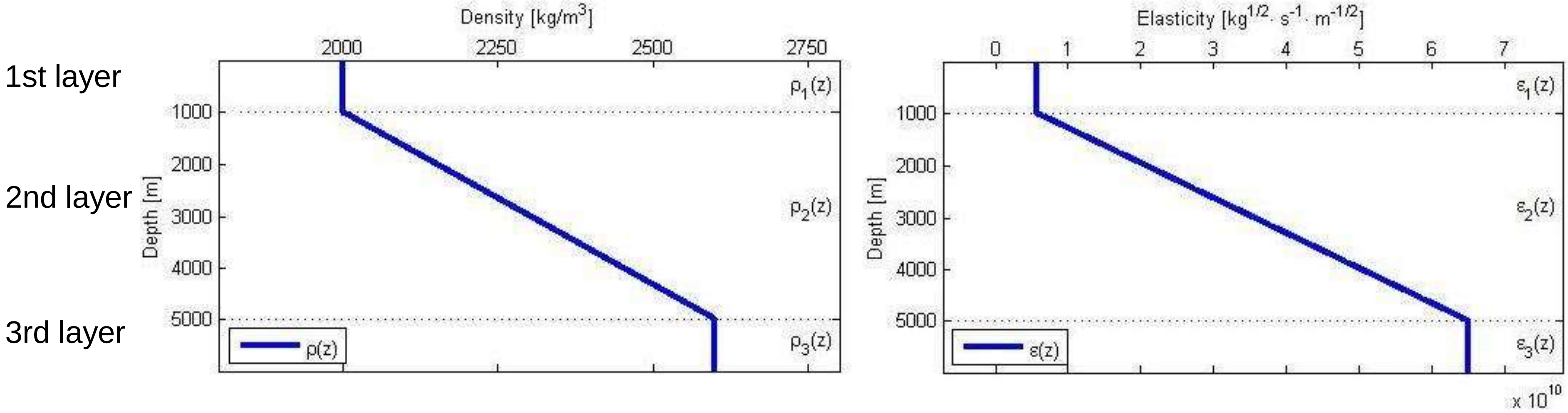
Balatonföldvár, 2014.03. 28. Meeting of Young Geoscientists

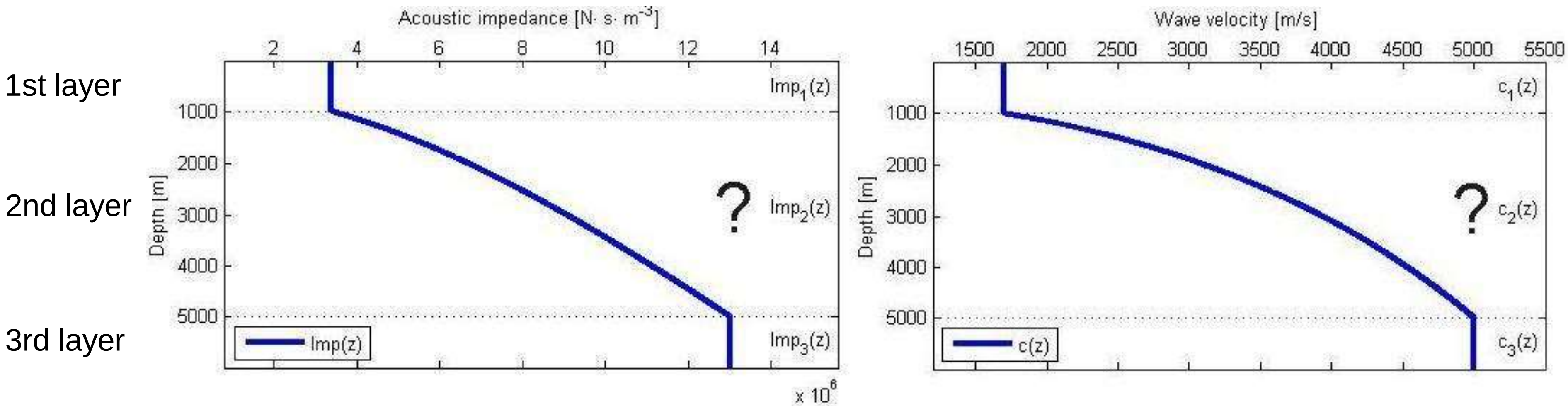
Outline

The model



- The model
  - top layer is isotropic and homogeneous: $\rho_1, \epsilon_1 = \lambda_1 + 2\mu_1$
  - middle layer is isotropic and **inhomogeneous**: $\rho_2(z), \epsilon_2(z) = \lambda_2(z) + 2\mu_2(z)$
  - bottom layer is isotropic and homogeneous: $\rho_3, \epsilon_3 = \lambda_3 + 2\mu_3$
  - Welded, plain boundaries at $z = z_1$ and $z = z_2$
- Excitation in the top layer
  - At t =0 we prescribe a propagating, arbitrary, longitudinal plain wave towards to the lower layers
  - The wavefronts of the plain wave are parallel to the plain boundaries

What we expect?



- Acoustic impedance and wave velocity of top and bottom layer
  - $c_i = \sqrt{\frac{\epsilon_i}{\rho_i}}$; $Imp_i = \rho_i \cdot c_i = \sqrt{\rho_i \cdot \epsilon_i}$; $i = 1,3$
- The middle layer
  - How to change the wave velocity with depth? $c_2(z) =^{?} \sqrt{\frac{\epsilon_2(z)}{\rho_2(z)}}$
  - How to change the acoustic impedance with depth? $Imp_2(z) =^{?} \sqrt{\rho_2(z) \cdot \epsilon_2(z)}$
- What reflections and the transmitted waves do we get?
  - Do we get permanently reflected waves from the middle layer?
- How the plain waves propagate in the middle layer?

Physical background

- Seismic wave equation for (λ, μ) isotropic media
  - $\rho \partial_t^2 \vec{u} = \nabla\lambda(\nabla \cdot \vec{u}) + [\nabla\vec{u} + (\nabla\vec{u})^T](\nabla\mu) + (\lambda + 2\mu)\nabla(\nabla\vec{u}) - \mu\nabla \times (\nabla \times \vec{u}) + \vec{f}$

- Density and Lamé parameters profiles in the model
  - $\rho(z) = \rho_1 \mathcal{H}(z_1 - z) + \rho_2(z)[\mathcal{H}(z - z_2) - \mathcal{H}(z - z_1)] + \rho_3 \mathcal{H}(z - z_2)$
  - $\lambda(z) = \lambda_1 \mathcal{H}(z_1 - z) + \lambda_2(z)[\mathcal{H}(z - z_2) - \mathcal{H}(z - z_1)] + \lambda_3 \mathcal{H}(z - z_2)$
  - $\mu(z) = \mu_1 \mathcal{H}(z_1 - z) + \mu_2(z)[\mathcal{H}(z - z_2) - \mathcal{H}(z - z_1)] + \mu_3 \mathcal{H}(z - z_2)$

- Longitudinal, arbitrary plain wave excitation
  - Plain wave excitation into $+z$ direction $\Rightarrow$ one spatial dimension $\Rightarrow \partial_x = \partial_y = 0$
  - Longitudinal wave excitation $\Rightarrow \vec{u} = \left(u_x(t,z), u_y(t,z), u_z(t,z)\right)^T = \left(0, 0, u_z(t,z)\right)^T$

Physical background

- The top layer

$$\begin{cases} \rho_1 \partial_t^2 u_{1z}(t,z) - \epsilon_1 \partial_z^2 u_{1z}(t,z) = 0 & \text{for} \quad t>0, \quad z<z_1 \\ u_{1z}(t=0,z) = Q\left(\frac{-z}{c_1}\right), \quad \partial_t u_{1z}(t,z)|_{t=0} = Q^{(1)}\left(\frac{-z}{c_1}\right) & \text{for} \quad z<z_1 \end{cases}$$

- The middle layer

$$\begin{cases} \rho_2(z)\partial_t^2 u_{2z}(t,z) - \epsilon_2(z)\partial_z^2 u_{2z}(t,z) - \partial_z\epsilon_2(z)\partial_z u_{2z}(t,z) = 0 & \text{for} \quad t>0, \quad z_1<z<z_2 \\ u_{2z}(t=0,z)=0, \quad \partial_t u_{2z}(t,z)|_{t=0} = 0 & \text{for} \quad z_1<z<z_2 \end{cases}$$

- The bottom layer

$$\begin{cases} \rho_3 \partial_t^2 u_{3z}(t,z) - \epsilon_3 \partial_z^2 u_{3z}(t,z) = 0 & \text{for} \quad t>0, \quad z_2<z \\ u_{1z}(t=0,z)=0, \quad \partial_t u_{3z}(t,z)|_{3=0} = 0 & \text{for} \quad z_2<z \end{cases}$$

- The boundary/conjugation conditions (continuity of the displacements and stresses)

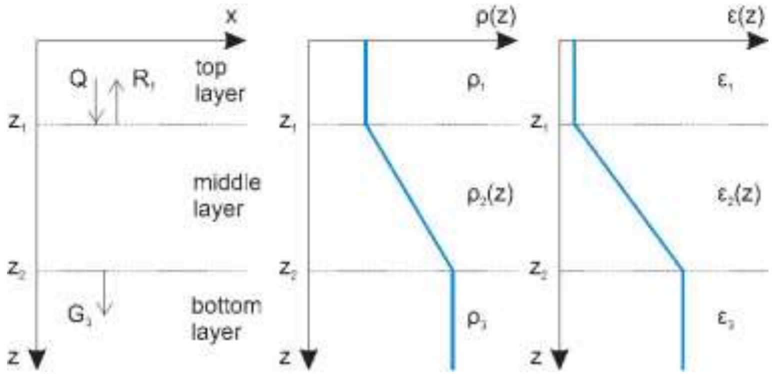
$$\begin{cases} u_{1z}(t,z=z_1) = u_{2z}(t,z=z_1) \\ \epsilon_1\partial_z u_{1z}(t,z)|_{z=z_1} = \epsilon_2(z=z_1)\partial_z u_{2z}(t,z)|_{z=z_1} \end{cases} \qquad \begin{cases} u_{2z}(t,z=z_2) = u_{3z}(t,z=z_2) \\ \epsilon_2(z=z_2)\partial_z 2z(t,z)|_{z=z_2} = \epsilon_3\partial_z u_{3z}(t,z)|_{z=z_2} \end{cases}$$

- Abbrevations

$$c_1 = \sqrt{\frac{\epsilon_1}{\rho_1}}, \quad c_3 = \sqrt{\frac{\epsilon_3}{\rho_2}}, \quad \rho_2(z) = Cz + D, \quad \epsilon_2(z) = Az + B$$

Part solutions

- General solution of the 1st and the 3rd layer
  - $u_i(t,z) = G_i\left(t - \frac{z}{c_i}\right) + R_i\left(t + \frac{z}{c_i}\right);$ $i = 1,3$
- Argument study



- The part solution of the top layer

$$u_1(t,z) = \begin{cases} Q\left(t - \frac{z}{c_1}\right) + R_1\left(t + \frac{z}{c_1}\right) & \text{if} \quad t + \frac{z - z_1}{c_1} > 0 \\ Q\left(t - \frac{z}{c_1}\right) & \text{if} \quad t + \frac{z - z_1}{c_1} < 0 \end{cases}$$

$$\text{for} \quad t > 0,\ z < z_1$$

- The part solution of the bottom layer

$$u_3(t,z) = \begin{cases} G_3\left(t - \frac{z}{c_3}\right) & \text{if} \quad t - \frac{z - z_2}{c_3} > 0 \\ 0 & \text{if} \quad t - \frac{z - z_2}{c_3} < 0 \end{cases}$$

$$\text{for} \quad t > 0,\ z > z_2$$

Part solutions

- The general solution of the middle layer with no restriction for (t, z)

$$
\begin{aligned}
u_2(s,z) &= G_2(t) *_t \mathcal{F}^{-1}\{U[a(s),1,x(s,z)]\}\left(t-\frac{z}{c_2}\right) + \Re_2(t) *_t \mathcal{F}^{-1}\{M[a(s),1,x(s,z)]\}\left(t-\frac{z}{c_2}\right) \\
&= G_2\left(t-\frac{z}{c_2}\right) *_t \mathcal{F}^{-1}\{U[a(s),1,x(s,z)]\}(t) + \Re_2\left(t-\frac{z}{c_2}\right) *_t \mathcal{F}^{-1}\{M[a(s),1,x(s,z)]\}(t) \\
&= G_2(t) *_t \mathcal{F}^{-1}\{U[a(s),1,x(s,z)]\}\left(t-\frac{z}{c_2}\right) + R_2(t) *_t \mathcal{F}^{-1}\{M[1-a(s),1,-x(s,z)]\}\left(t+\frac{z}{c_2}\right) \\
&= G_2\left(t-\frac{z}{c_2}\right) *_t \mathcal{F}^{-1}\{U[a(s),1,x(s,z)]\}(t) + R_2\left(t+\frac{z}{c_2}\right) *_t \mathcal{F}^{-1}\{M[1-a(s),1,-x(s,z)]\}(t),
\end{aligned}
$$

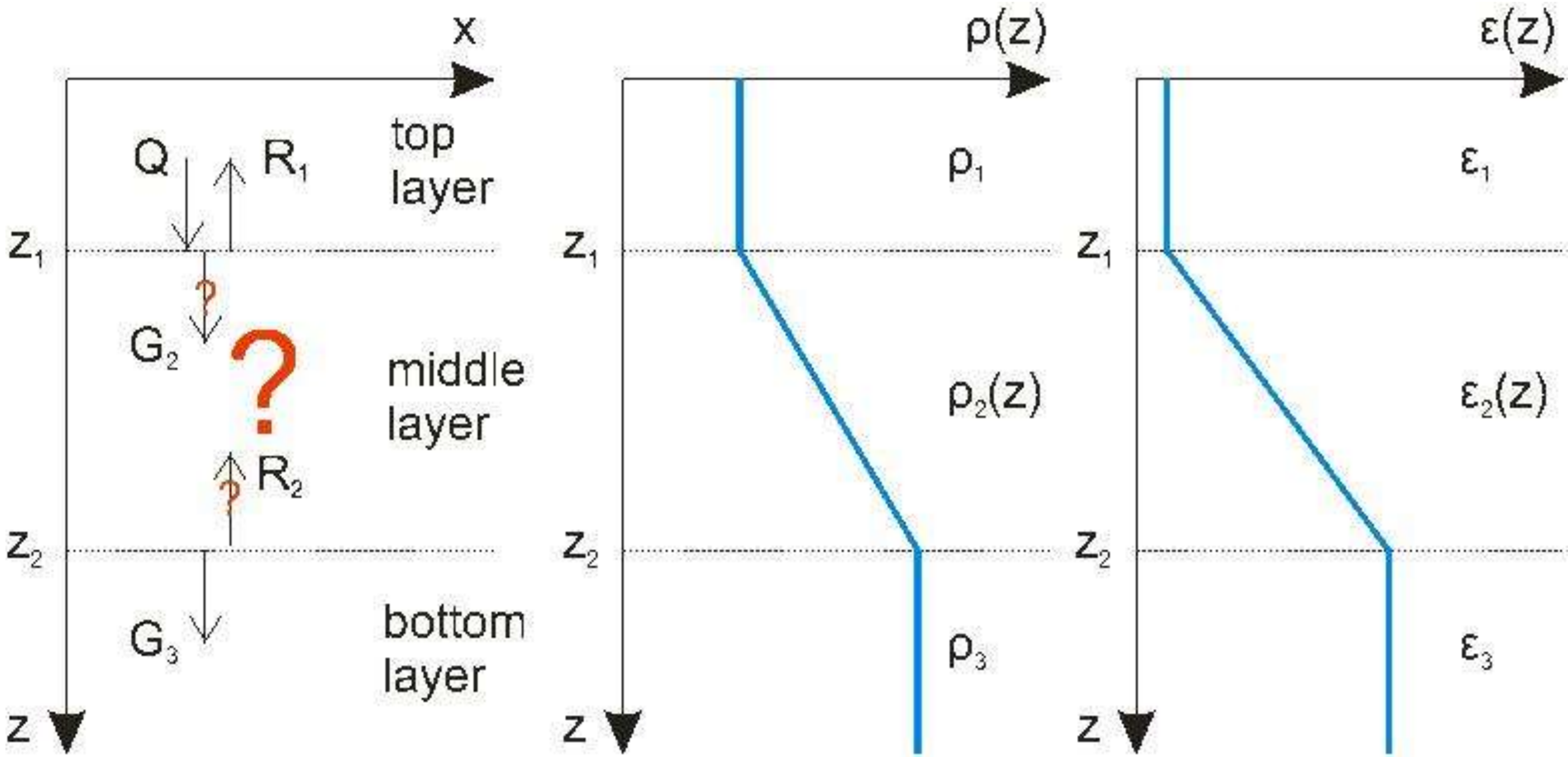
where $M[a,b,x]$, $U[a,b,x]$ is the confluent hypergeometric functions of the first and second kind, respectively, G_2, $R_2/\Re_2$ are unknowns functions, and

$$
c_2 = \sqrt{\frac{A}{C}}, \quad a(s) = \frac{1}{2} + j\pi\left(\frac{D}{C} - \frac{B}{A}\right)\frac{s}{c_2}, \quad x(s,z) = j4\pi\frac{z}{c_2}s + j4\pi\frac{B}{A}\frac{s}{c_2}.
$$

- The part solution of the the middle layer (for $t > 0$, $z_1 < z < z_2$) ???

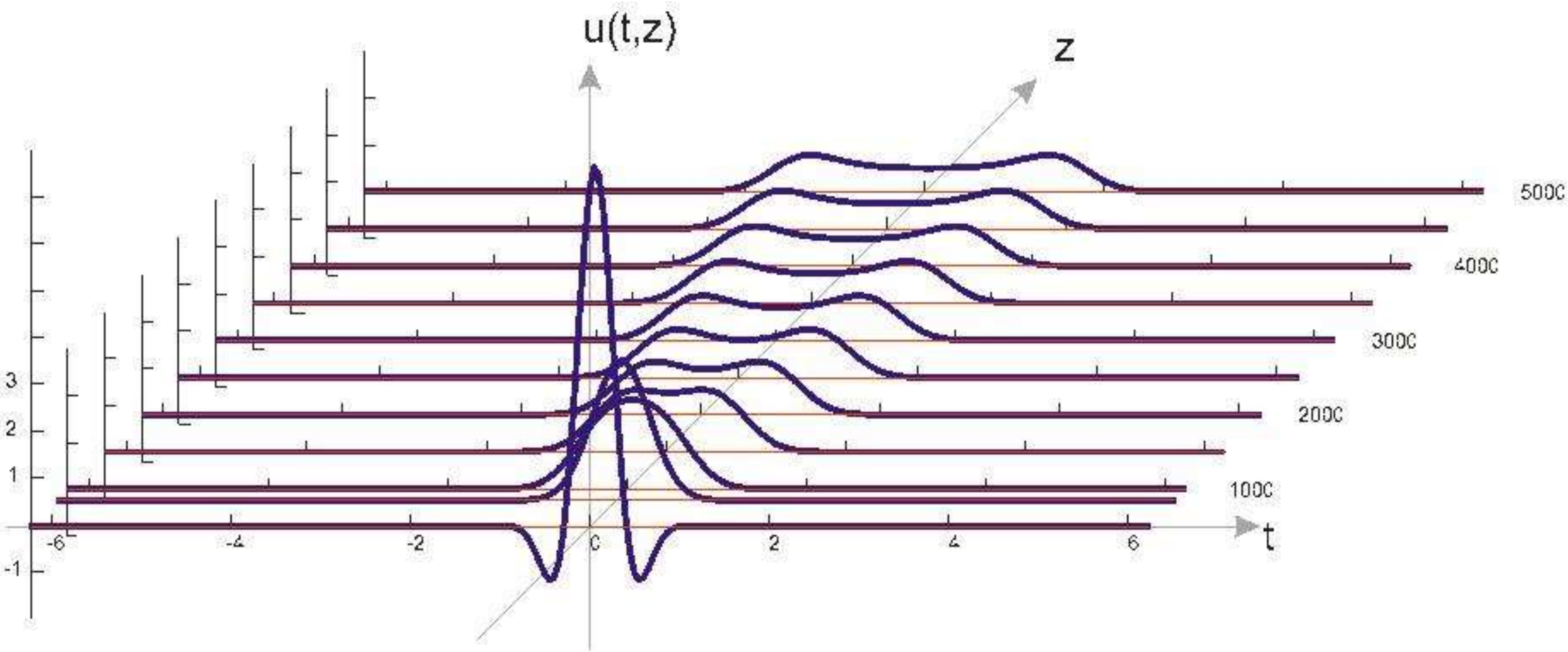
Conjugation condition

- Boundary/conjugation conditions
  - at each boundaries we have two conjugation equations
    - continuity of the displacements at $z = z_1$, and $z = z_2$
    - continuity of the stresses at $z = z_1$, and $z = z_2$



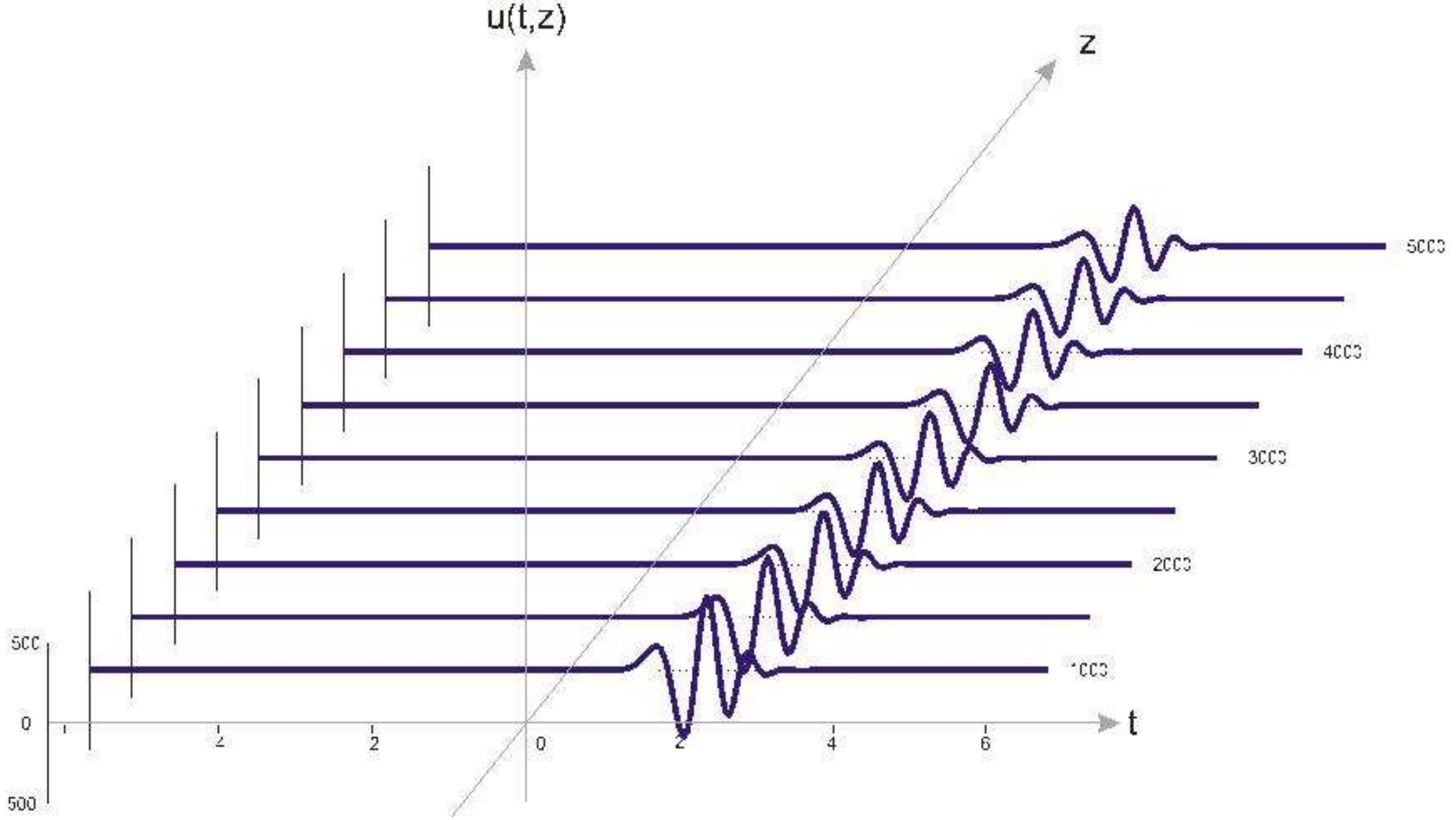
Analysis of the general solution of the middle layer

- Confluent hypergeometric function of the first kind
  - $u_2(t,z) = R_2(t) *_t \mathcal{F}^{-1}\{M[a(s),1,x(s,z)]\}\left(t - \frac{z}{c_2}\right)$
  - $R_2(t) = 3 \cdot e^{-5 \cdot t^2}$



Analysis of the general solution of the middle layer

- Confluent hypergeometric function of the second kind
  - $u_2(t,z) = G_2(t) *_t \mathcal{F}^{-1}\{U[a(s),1,x(s,z)]\}\left(t - \frac{z}{c_2}\right)$
  - $G_2(t) = 3 \cdot e^{-5 \cdot t^2}$



Thank you for your attention!